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$=-\int_{\frac{1}{2}\pi}^{\frac{1}{4}\pi}\sin\phi d\phi=\int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi}\sin\phi d\phi$. Therefore it is evident $\int_0^{\frac{1}{4}\pi}\log\sin(\frac{1}{4}\pi+x)dx$

$-\int_0^{\frac{1}{4}\pi}\cos x dx=0$.

74. Proposed by EDWARD R. ROBBINS, A. B., Mathematical Master in the Lawrenceville School, Lawrenceville, N. J.

A circular ring, whose radii are a and b, is cut by a plane making the area of the section (or sections) a maximum. Required the position of the plane, and the nature and area of the section (or sections).

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Science, Chester High School, Chester, Pa.

Let the axis of the surface be taken as the axis of z, then the equation to the ring is

$$(x^2+y^2+z^2+ab)^2=(a+b)^2(x^2+y^2).$$

The maximum section will be made by a plane passing through the center of the ring and making an angle β such that $\sin\beta<[(a-b)/(a+b)]$.

Let $z=x\tan\beta$ be this plane.

$\therefore (x^2\sec^2\beta+y^2+ab)^2=(a+b)^2(x^2+y^2)$.

The area of this section is $(\pi/\sec^2\beta)(a^2-b^2)$.

This is a maximum when $\sec^2\beta$ is least, $\therefore \beta=0$.

$\therefore (x^2+y^2+ab)^2=(a+b)^2(x^2+y^2)$.

$\therefore$ the plane coincides with the xy plane and divides the ring into two equal parts. The section is included between the circles $x^2+y^2=a^2$ and $x^2+y^2=b^2$, area$=\pi(a^2-b^2)$.

MECHANICS.

64. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

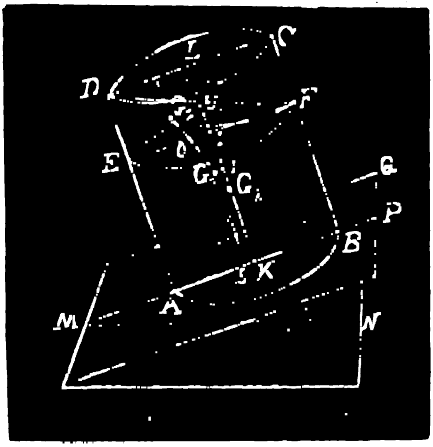
A cylindrical vessel, radius of vessel r and altitude h, is filled with water and rests on a horizontal plane. It is required to ascertain the maximum angle of elevation to which the plane may be raised without the vessel falling, allowing the coefficient of friction to be such as to prevent sliding, and the water to overflow as the plane is raised.

I. Solution by the PROPOSER.

Let MQ be the inclined plane; $ABCD$ a vertical section of the vessel; $ABFD$ a vertical section of the water when the vessel is on the point of turning over; $AB=2R$, the diameter of the vessel; $AD=H$, its altitude; and the angle $PMN=\theta$, maximum inclination of the plane. The quantity of water in the vessel at the time the vessel is on the point of turning over consists of the cylindrical part whose vertical section is $ABFE$ and the cylindrical ungula whose vertical section is EFD. Let V_2 be the volume of the cylindrical part, and V_1 the volume of the ungula. Let G_2 be the center of gravity of the cylindrical part, G_1 the center of gravity of the ungula, and G_0 the common center of gravity of both.

The vessel will be on the point of turning over when a vertical line thro' G_0 passes through A. Hence, the condition for the maximum inclination is

$$\frac{SA}{G_0 S}=\tan\theta.$$



G_2 is on the axis of the cylinder and midway between K and I. Now $KI=KL-IL(=DE)=H-2R\tan\theta$. Hence $KG_2=\frac{1}{2}H-R\tan\theta$.

The center of gravity of the ungula is found as follows: Let FE be the positive x-axis, FC the positive z-axis, and a line perpendicular to these two lines at F the y-axis. Let $\bar{x}$, $\bar{y}$, and $\bar{z}$ be the coördinates of the center of gravity of the ungula. Since the plane EFD is a plane of symmetry, $\bar{y}=0$.

$$\bar{x}=\frac{\int_0^{2R}\int_0^{\sqrt{(2Rx-x^2)}}\int_0^{x\tan\theta} x\,dx\,dy\,dz}{\int_0^{2R}\int_0^{\sqrt{(2Rx-x^2)}}\int_0^{x\tan\theta} dx\,dy\,dz}=\frac{2}{\pi R^3\tan\theta}\int_0^{2R}\int_0^{\sqrt{(2Rx-x^2)}}\int_0^{x\tan\theta} x\,dx\,dy\,dz$$

$$=\frac{2}{\pi R^3}\int_0^{2R}\int_0^{\sqrt{(2Rx-x^2)}} x^2\,dx\,dy=\frac{2}{\pi R^3}\int_0^{2R} x^2(2Rx-x^2)^{\frac{1}{2}}\,dx,$$

$$=\frac{2}{\pi R^3}\left[\left(-\tfrac{1}{4}x(2Rx-x^2)^{\frac{3}{2}}\right)_0^{2R}+R\int_0^{2R} x\sqrt{(2Rx-x^2)}\,dx\right]$$

$$=\frac{2}{\pi R^2}\int_0^{2R} x\sqrt{(2Rx-x^2)}dx=\frac{2}{\pi R^2}\int_0^{2R}\left[R\sqrt{(2Rx-x^2)}-(R-x)\sqrt{(2Rx-x^2)}\right]dx$$

$$=\frac{2}{\pi R^2}\left[\frac{R(x-R)}{2}\sqrt{(2Rx-x^2}+\tfrac{1}{2}R^3\sin^{-1}\frac{x-R}{R}-\tfrac{1}{3}(2Rx-x^2)^{\frac{3}{2}}\right]_0^{2R}=\tfrac{5}{4}R=FO\,;$$

$$\bar{z}=\frac{\int_0^{2R}\int_0^{\sqrt{(2Rx-x^2)}}\int_0^{x\tan\theta} z\,dx\,dy\,dz}{\int_0^{2R}\int_0^{\sqrt{(\frac{1}{2}Rx-x^2)}}\int_0^{x\tan\theta} dx\,dy\,dz}=\tfrac{5}{8}R\tan\theta=G_1 O.$$

Hence, $G_1 U=\frac{5}{4}R-R=\frac{1}{4}R$; $G_2 U=G_2 I+IU=\frac{1}{2}H-R\tan\theta+\frac{5}{8}R\tan\theta$

$=\frac{1}{8}(4H-3R\tan\theta)$; and $G_1G_2=\surd(G_1U^2+G_2U^2)=\frac{1}{8}\surd[(4H-3R\tan\theta)^2+4R^2]$.

Taking moments about G_0, we have $G_1G_0\times V_1=G_0G_2\times V_2$, the masses being proportional to the volumes. $\therefore G_1G_0 : G_0G_2=V_2 : V_1$, or $G_1G_2 : G_0G_2 =V_2+V_1 : V_1$, or $\frac{1}{8}\surd[4H-3R\tan\theta)^2+4R^2] : G_0G_2=\pi R^2(H-R\tan\theta) : \pi R^3\tan\theta$.

$\therefore G_0G_2=R\tan\theta\surd[(4H-3R\tan\theta)^2+4R^2]/8(H-R\tan\theta)$.

$G_1U : G_1G_2=G_0J(=SK) : G_0G_1$, whence $G_0J=SK=R^2\tan\theta/4(H-R\tan\theta)$.

$$\therefore SA=R-\frac{R^2\tan\theta}{4(H-R\tan\theta)}=\frac{R(4H-5R\tan\theta)}{4(H-R\tan\theta)}.$$

$$G_1U : UG_2=G_0J : JG_2,\text{ whence } JG_2=\frac{R\tan\theta(4H-3R\tan\theta)}{8(H-R\tan\theta)}.$$

$$\therefore JK=G_0S=\frac{R(4H-3R\tan\theta)\tan\theta}{8(H-R\tan\theta)}+(\tfrac{1}{2}H-R\tan\theta)$$

$$=\frac{R(4H-3R\tan\theta)\tan\theta+8(\frac{1}{2}H-R\tan\theta)(H-R\tan\theta)}{8(H-R\tan\theta)}$$

$$\frac{SA}{G_0S}=\tan\theta=\frac{R(4H-5R\tan\theta)}{4(H-R\tan\theta)}\Big/\frac{R(4H-3R\tan\theta)\tan\theta+8(\frac{1}{2}H-R\tan\theta)(H-R\tan\theta)}{8(H-R\tan\theta)}$$

$$\text{or }\tan\theta=\frac{2R(4H-5R\tan\theta)}{R(4H-3R\tan\theta)\tan\theta+(4H-8R\tan\theta)(H-R\tan\theta)},$$

whence $5R^2\tan^3\theta-8RH\tan^2\theta+2(2H^2+R^2)\tan\theta-8RH=0$, an equation from which $\tan\theta$ can be found, and, therefore, θ.

II. Solution by J. SCHEFFER, A. M., Hagerstown, Md.

Let the required maximum angle be denoted by θ. $ABCD$ represents the vessel, $ABFD$ the quantity of water, and G_0 the center of gravity of the portion $ABFD$. Let $SA=x$ be any portion of the diameter AB, and x_0 and z_0 be the coordinates of the center of gravity G_0, in the direction of AB and AC, and V the volume of $ABFD$. Then, since $CF=2r\tan\theta$, and SQ' (Q' does not appear in the above figure. It is the intersection of SG_0 and $DF)=h-x\tan\theta$,

$$V_{x_0}=2\int_0^{2x}x(h-x\tan\theta)\surd(2rx-x^2)dx=\frac{\pi r^3}{8}(4h-5hx\tan\theta).$$

$$V_{z_0}=\int_0^{2r}(h-x\tan\theta)\surd(2rx-x^2)dx=\frac{\pi r^2}{8}(8h^2--8hr\tan\theta+5r^2\tan^2\theta).$$

The limiting position of equilibrium is that for which the vertical from G_0 will meet A, or $\tan\theta=x_0/z_0$.

$\therefore$ We obtain the final equation

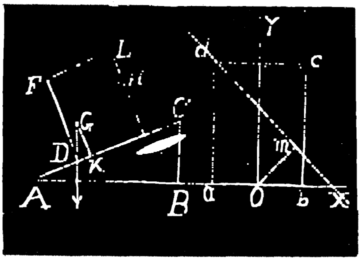
$$5r^2\tan^3\theta-8hr\tan^2\theta+2(2h^2+5r^2)\tan\theta-8hr=0.$$

III. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Science, Chester High School, Chester, Pa.

First find the center of gravity of the portions of the cylinder between ab and dm. Let $\angle cdm=\beta$, $ab=2r$, $bc=h$. Then equation to the cylinder is $x^2+z^2=r^2$; of line dm, $x\sin\beta+y\cos\beta=h\cos\beta-r\sin\beta$.

Let (l, k) be the coördinates of the center of gravity.

The limits of x are $-r$ and $+r$; of y, 0 and $[h-(r+x)\tan\beta]=y'$; of z, 0 and $\sqrt{(r^2-x^2)}=z'$.



$$\therefore\ l=\frac{\int_{-r}^{r}\int_{0}^{y'}\int_{0}^{z'}xdxdydz}{\int_{-r}^{r}\int_{0}^{y'}\int_{0}^{z'}dxdydz}=\int_{-r}^{r}xy'z'dx\Big/\int_{-r}^{r}y'z'dx$$

$$k=\frac{\int_{-r}^{r}\int_{0}^{y'}\int_{0}^{z'}ydxdydz}{\int_{-r}^{r}\int_{0}^{y'}\int_{0}^{z'}dxdydz}=\tfrac{1}{2}\int_{-r}^{r}y'^2z'dx\Big/\int_{-r}^{r}y'z'dx.$$

$$\therefore\ l=-\frac{r^2\tan\beta}{4(h-r\tan\beta)},\quad \frac{4h^2-8hr\tan\beta+5r^2\tan^2\beta}{8(h-r\tan\beta)}=k.$$

Now $LFH=DGK=CAB=\beta$.

$$\therefore\ \tan\beta=DK/GK=\frac{r-l}{k}=\frac{8hr-10r^2\tan\beta}{4h^2-8hr\tan\beta+5r^2\tan^2\beta}.$$

Let $u=\tan\beta$.

$$\therefore\ u^3-\frac{8h}{5r}u^2+\frac{4h^2+10r^2}{5r^2}u=\frac{8h}{5r}.$$

Let $u=v+(8h/15r)$.

$$\therefore\ v^3+\frac{150r^2-4h^2}{75r^2}v+\frac{3246h^3-1800hr^2}{3375r^3}=0.$$

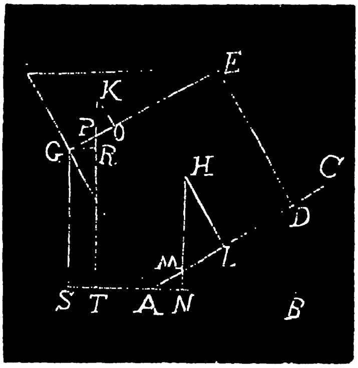
Let $\dfrac{150r^2-4h^2}{75r^2}=3m$, and $\dfrac{3246h^3-1800hr^2}{3375r^3}=2n$.

$\therefore\ v^3+3mv+2n=0.\quad v=[-n+\sqrt{(n^2+m^3)}]^{\frac{1}{3}}+[-n-\sqrt{(n^2+m^3)}]^{\frac{1}{3}}$.

In the above, the weight of the vessel is disregarded.

IV. Solution by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, University P. O., Miss.

Let AC be the inclined plane making an angle β with the horizontal AB, $ADEF$ the water in the vessel, EG parallel to AD, H and K the centers of gravity of the parts $ADEG$ and EGF respectively. The construction of the rest of the figure is evident.



Denote AF, AG, and AD by h, $2a$, and $2r$, respectively. Neglecting the weight of the vessel, it will be on the point of overturning when $ADEG\times AN=EFG\times AT$......(1).

To find the center of gravity of EFG, consider EG the positive direction of the x-axis, ED the negative direction of the z-axis, and a perpendicular to these the y-axis. The solid question is bounded by the cylindrical surface $y^2=2rx-x^2$, and the planes $z=0$, $z=\tan\beta.x$. Since the xz-plane is one of symmetry, $\bar{y}=0$.

$$\bar{x}=\frac{\int_0^{2r}\int_0^{y_1}\int_0^{z_1}xdxdydz}{\int_0^{2r}\int_0^{y_1}\int_0^{z_1}dxdydz},\quad \bar{z}=\frac{\int_0^{2r}\int_0^{y_1}\int_0^{z_1}zdxdydz}{\int_0^{2r}\int_0^{y_1}\int_0^{z_1}dxdydz},$$

where $z_1=\tan\beta.x$, $y_1=\sqrt{(2rx-x^2)}$.

From these, $\bar{x}=\frac{5}{4}r=EO$, $\bar{z}=\frac{5}{8}r\tan\beta=OK$.

$OP=OK\tan\beta=\frac{5}{8}r\tan^2\beta$;

$GP=GE-OE--OP=\frac{3}{4}r-\frac{5}{8}r\tan^2\beta$;

$GR=GP\cos\beta=\frac{3}{4}r\cos\beta-\frac{5}{8}r\cos\beta\tan^2\beta$;

$SA=AG\sin\beta=2a\sin\beta$;

$AT=SA-GR=2a\sin\beta-\frac{3}{4}r\cos\beta+\frac{5}{8}r\cos\beta\tan^2\beta$.

$ML=HL\tan\beta=a\tan\beta$;

$AM=AL-ML=r-a\tan\beta$;

$AN=AM\cos\beta=r\cos\beta-a\sin\beta$.

Volume of $ADEG=2\pi ar^2$.

Volume of $EGF=2\int_0^{2r}\int_0^{\sqrt{(2rx-x^2)}}\int_0^{x\tan\beta}dxdydz=\pi r^3\tan\beta$.

[This result comes easily without the calculus].

Substituting in equation (1), transposing, dividing through by $\cos\beta$, rearranging, and clearing of fractions,

$$5r^2\tan^3\beta+16ar\tan^2\beta+2(8a^2-3r^2)\tan\beta-16ar=0.$$

Substituting in this, $\frac{1}{2}h-r\tan\beta$ for a, and reducing,

$$5r^2\tan^3\beta-8rh\tan^2\beta+2(5r^2+2h^2)\tan\beta-8rh=0,$$

from which β may be found.

AVERAGE AND PROBABILITY.

62. Proposed by O. S. KIBLER, Superintendent of Schools, Middleburg, O.

A bag contains any number of balls, which are equally likely to be white or black; one is drawn and found to be white. Show that the chance of drawing another white one, the first ball not being replaced, is two-thirds. [From *C. Smith's Treatise on Algebra*, page 615].

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Science, Chester High School, Chester, Pa.

If m balls are drawn and turn out white, the chance that n others drawn will be white is: $p=[(m+1)/(m+n+1)]$. In the problem, $m=n=1$.

$\therefore p=\frac{2}{3}$. Otherwise, $p=\int_0^1 x^2dx/\int_0^1 xdx=\frac{2}{3}$.

This is the simplest case of the article on page 107, No. 4, Vol. II. of the MONTHLY.

64. Proposed by REV. W. A. WHITWORTH, A. M.

O is a given point within a triangle; P is a random point within the same. The line through O and P is produced so as to divide the triangle into a trapezium and a triangle. Find the average area of this triangle. [From the *Educational Times*, London, Eng.]

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Science, Chester High School, Chester, Pa.

Let $\triangle$ be the average area required, $\triangle_1$ the average area of BFG; (m, n) coördinates of O. Then the coördinates of E are $(b, bn/m)$; of D, $[b^2n/(ab-am+bn), abn/(ab-am+bn)]$.

Area $ABC=\frac{1}{2}ab\sin C$.

Area $ACO=\frac{1}{2}bn\sin C$.

Area $ADO=bn(am-bn)\sin C/[2(ab-am+bn)]$.

Area $COE=bn(b-m)\sin C/(2m)$.

Area $DOEB=ABC-(ACO+ADO+COE)$.